

Flat - brane compactifications in Supergravity induced by scalars

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ABSTRACT: We discuss flat compactifications of supergravities in diverse dimensions in the presence of branes. The compactification is induced by the scalar fields of supergravity and it is such that there is no relic cosmological constant on the brane, rendering this way the latter flat. We discuss in particular the $D = 4$, $\mathcal{N} = 2, 4$ and $D = 8$, $\mathcal{N} = 1$ supergravities with $n = 1, 2, 3$ vector multiplets where the scalar manifolds are Grassmannian cosets of the form $SO(2, n)/SO(2) \times SO(n)$. By introducing branes at certain points in the transverse space, finite energy solutions to the field equations are constructed. Some of the solutions we present may be interpreted as intersecting branes.

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1. Introduction

Codimension-two brane solutions of gravitational theories have attracted much interest the recent years. The distinguishing features of such solutions are: (i) the brane worldvolume is always Ricci-flat, irrespective of any vacuum energy induced on the brane, and (ii) the internal space has the same local geometry as it would have in the absence of branes, apart from conical deficit angles proportional to the brane tensions [1]-[16]. The first property motivates the study of such solutions in the context of six-dimensional theories, since the corresponding Ricci-flat 3-branes provide a new perspective for a possible solution of the cosmological constant problem [17]-[23]. More generally, codimension-two brane solutions can be examined in gravitational theories in diverse dimensions, where they correspond to defects of dimension lower or higher than four.

The codimension-two solutions mentioned above may be triggered by matter fields appearing in the theory, such as p -form gauge fields [2, 5, 6, 7] or, most importantly, by scalar fields. On the other hand, for the case of sigma models with a compact target space, solutions of this type have been found in [12, 13]; however, sigma models with such scalar manifolds do not occur in supergravity. For the case of non-compact sigma models, there are two prototype solutions. The first type of solutions [10] generalize the “teardrop” solution of [24] to account for the presence of branes; here the internal 2-dimensional manifold is a *non-compact* space of finite volume [25, 26, 27] and the geometry has a naked singularity at its boundary which, however, may be rendered harmless by imposing appropriate boundary conditions. These boundary conditions guarantee that the conservation laws of the theory are not spoiled and energy, momentum angular momentum etc do not “leak” from the boundary. The second type of solutions are based on the “stringy cosmic string” of [28]. In this case, the internal geometry can be non-singular provided that the brane tensions are restricted to a certain range, and, in fact, correspond to a compact manifold of Euler number 2 provided that the brane tensions are appropriately fine-tuned [29]. Moreover, the existence of modular symmetries in the non-compact case guarantees that the scalars and the metric are actually single-valued, unlike the compact case where this issue is not clear.

In this paper we present codimension-two solutions of supergravity models in diverse dimensions, in the presence of branes. In particular, we consider D -dimensional supergravity theories coupled to nonlinear sigma models, with the sigma-model target spaces being non-compact Kähler manifolds. We seek exact solutions of the form $M^{D-n} \times \mathcal{K}$, where M^{D-n} is a flat Minkowski space and \mathcal{K} is an n -dimensional internal space. As concrete examples, we consider the cases of $\mathcal{N} = 4$ and $\mathcal{N} = 2$ supergravity in 4 dimensions (with the solutions corresponding to strings), as well as the cases of minimal supergravities coupled to vector multiplets in 8 dimensions (with the solutions corresponding to parallel or intersecting five-branes). For all of the above cases, the scalar manifold is special Kähler of the form $\frac{SL(2, \mathbb{R})}{U(1)} \times \frac{SO(2, n)}{SO(2) \times SO(n)}$ [30, 31, 32] or a Grassmannian coset $\frac{SO(2, n)}{SO(2) \times SO(n)}$ [33]. To find the explicit solutions, we follow the guidelines of [28], employing a holomorphic ansatz for the scalars that restricts the latter to lie in the fundamental domain of the modular groups and allowing modular $SL(2, \mathbb{Z})$, $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ or $Sp(4, \mathbb{Z})$ jumps around certain points in the internal space. This leads to scalar field configurations with finite energy per unit volume. Explicit solutions may be presented only if $n \leq 3$, since it is only for these cases that the modular forms required to construct the solutions are explicitly known. The solutions under consideration generically possess singularities appearing in the form of conical deficit angles; in order to arrive at non-singular solutions, one has to arrange the total deficit angle to be 4π in which case the internal space compactifies to \mathbf{S}^2 .

The solutions described above can be generalized to include brane configurations as well. In this case, the branes are introduced at the points where the scalar fields diverge leading to extra delta-function contributions to the scalar energy-momentum tensor. The branes induces further deficit angles, proportional to their tensions in the internal space. In such a scenario, the requirement for the absence of conical singularities may be fulfilled by suitably tuning[1, 12, 29] the brane tensions so that the total deficit angle equals 4π .

The structure of this paper is as follows. In section 2, we review the solution of the equations of motion for a gravitational theory coupled to a Kähler sigma model, we examine the prototype stringy cosmic string solution, and we also comment of the issue of codimension-four solutions. In section 3, we describe the essential aspects of the Kähler sigma models under consideration and we state the explicit form of the Kähler potentials in terms of the supergravity fields. In section 4, we apply the above results to the cases $\mathcal{N} = 4$ and $\mathcal{N} = 2$ supergravity in 4 dimensions and we present the corresponding “stringy cosmic string” solutions, while we also consider the case of minimal supergravity in 8 dimensions and we present the corresponding 5-brane solution as well as a four-dimensional intersecting-brane solution. Finally, in section 5 we summarize our main results.

2. Kähler sigma models

Our general setup corresponds to a D -dimensional theory of gravity coupled to a Kähler sigma model. This is a generic situation in almost all supergravity theories in four or higher dimensions. Without specifying the scalar sigma model in detail at this stage, we assume that its target space is a Kähler manifold \mathcal{M} spanned by the complex coordinates $(\varphi^i, \bar{\varphi}^{\bar{j}})$ and characterized by the Kähler potential $K(\varphi^i, \bar{\varphi}^{\bar{j}})$ and the metric $K_{i\bar{j}}(\varphi^i, \bar{\varphi}^{\bar{j}}) = \partial_i \partial_{\bar{j}} K(\varphi^i, \bar{\varphi}^{\bar{j}})$. The dynamics of this system is described by the action

$$S = \int d^D x \sqrt{-g} M_*^{D-2} \left(\frac{1}{2} R - K_{i\bar{j}}(\varphi^i, \bar{\varphi}^{\bar{j}}) \partial_M \varphi^i \partial^M \bar{\varphi}^{\bar{j}} \right) , \quad (2.1)$$

where M_*^{D-2} is the D -dimensional Planck mass. The equations of motion as follow from (2.1) are the scalar equation of motion

$$\frac{1}{\sqrt{-g}} \partial_M (\sqrt{-g} K_{i\bar{j}} \partial^M \varphi^i) - \partial_{\bar{j}} K_{i\bar{k}} \partial_M \varphi^i \partial^M \bar{\varphi}^{\bar{k}} = 0 , \quad (2.2)$$

and the Einstein equation

$$R_{MN} = K_{i\bar{j}} (\partial_M \varphi^i \partial_N \bar{\varphi}^{\bar{j}} + \partial_M \bar{\varphi}^{\bar{j}} \partial_N \varphi^i) . \quad (2.3)$$

We are looking for solutions of the form $M^{D-n} \times \mathcal{K}$, where M^{D-n} is a flat $(D-n)$ -dimensional Minkowski space-time parametrized by the coordinates x^μ and \mathcal{K} is an internal complex

manifold, parametrized by the complex coordinates $(z^a, \bar{z}^{\bar{b}})$ and metric $k_{a\bar{b}}(z^a, \bar{z}^{\bar{b}})$. To solve (2.2),(2.3), we assume that the scalars depend only on the internal coordinates, so that the solution we are after is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + k_{a\bar{b}}(z^a, \bar{z}^{\bar{b}}) dz^a d\bar{z}^{\bar{b}} , \quad \varphi^i = \varphi^i(z^a, \bar{z}^{\bar{b}}) . \quad (2.4)$$

Inserting this ansatz in Eqs. (2.2) and (2.3), and making use of standard Kähler identities such as $\partial_{[k} K_{i]\bar{j}} = 0$ and $\partial_{[c} k_{a]\bar{b}} = 0$, we find that these equations reduce to

$$k^{a\bar{b}} \partial_a \partial_{\bar{b}} \varphi^i + K^{i\bar{j}} \partial_k K_{l\bar{j}} k^{a\bar{b}} \partial_a \varphi^k \partial_{\bar{b}} \varphi^l = 0 , \quad (2.5)$$

and

$$\partial_a \partial_{\bar{b}} \ln \det k_{a\bar{b}} = -K_{i\bar{j}} (\partial_a \varphi^i \partial_{\bar{b}} \bar{\varphi}^{\bar{j}} + \partial_a \bar{\varphi}^{\bar{j}} \partial_{\bar{b}} \varphi^i) , \quad (2.6)$$

respectively. Starting from the scalar equation (2.5), we see that it is automatically satisfied when the φ^i are holomorphic or antiholomorphic functions. Restricting for definiteness to the holomorphic case,

$$\varphi^i = \varphi^i(z^a) , \quad (2.7)$$

we find that the Einstein equation (2.6) reduces to

$$\partial_a \partial_{\bar{b}} \ln \det k_{a\bar{b}} = -K_{i\bar{j}} \partial_a \varphi^i \partial_{\bar{b}} \bar{\varphi}^{\bar{j}} = -\partial_a \partial_{\bar{b}} K , \quad (2.8)$$

and hence it is solved by

$$\det k_{a\bar{b}}(z^a, \bar{z}^{\bar{b}}) = e^{-K(\varphi^i, \bar{\varphi}^{\bar{j}})} |F(z^a)|^2 , \quad (2.9)$$

where $F(z^a)$ is an arbitrary holomorphic function. Given a specific scalar manifold \mathcal{M} and a specific ansatz for the moduli $\varphi^i(z^a)$, the choice of $F(z^a)$ is dictated by the symmetries of the moduli space and by geometric properties such as the non-degeneracy of the metric and the absence of curvature singularities.

Generalizing our solution, we may also consider a $(D - n - 1)$ -brane located at $z^a = 0$ to which the scalar fields are not coupled. These branes contribute an additional energy momentum tensor to the right-hand side of Einstein equations of the form¹

$$T_{\mu\nu} = -g_{\mu\nu} T_0 \delta^{(n)}(z) \quad T_{ab} = T_{a\bar{b}} = T_{\bar{a}\bar{b}} = 0 \quad (2.10)$$

where $\mu, \nu = 0, \dots, D - n - 1$ and T_0 is the tension of the brane located at the origin. Then, (2.8) is changed to

$$\partial_a \partial_{\bar{b}} \ln \det k_{a\bar{b}} = -K_{i\bar{j}} \partial_a \varphi^i \partial_{\bar{b}} \bar{\varphi}^{\bar{j}} - k_{a\bar{b}} \frac{T_0}{M_*^{D-2}} \delta^{(n)}(z) , \quad (2.11)$$

which can be solved under certain conditions, as we will see below.

¹where $\int d^n z \sqrt{\det k_{a\bar{b}}} \delta^{(n)}(z) = 1$

2.1 Complex dimension one

We will first consider the case where the internal space has complex dimension one, which is relevant for seeking codimension-two brane solutions with an internal compact or non-compact space. Parametrizing the transverse space by the complex coordinate z , the explicit form of the solution (2.9) reads

$$\varphi^i = \varphi^i(z) , \quad ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + e^{-K} |F(z)|^2 dz d\bar{z} . \quad (2.12)$$

The energy density of such configurations can be written in the BPS-like form

$$E = \frac{i}{2} \int_{\mathcal{K}} d^2 z K_{i\bar{j}} \partial \varphi^i \bar{\partial} \varphi^{\bar{j}} = \frac{i}{2} \int_{\varphi(\mathcal{K})} \partial \bar{\partial} K , \quad (2.13)$$

where ∂ and $\bar{\partial}$ are Dolbeault operators and in the second integral, the domain of integration has been pulled back to the image $\varphi(\mathcal{K})$ of the internal manifold \mathcal{K} . In the presence of brane with tension T_0 , the above relations are modified to

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + e^{-K} |F(z)|^2 |z|^{-T_0/\pi M_*^{D-2}} dz d\bar{z} \quad (2.14)$$

and

$$E = \frac{i}{2} \int_{\mathcal{K}} d^2 z K_{i\bar{j}} \partial \varphi^i \bar{\partial} \varphi^{\bar{j}} + \frac{T_0}{M_*^{D-2}} . \quad (2.15)$$

Although the first form of the integral in (2.13) may appear to give a zero answer due to the fact that the integration domain is a compact surface, the Kähler potential is not globally well-defined on \mathcal{M} and hence the integral may contain jumps that render it nonzero. This is made more explicit in the second form of the integral in terms of the Kähler potential. Indeed, if there exist symmetries of \mathcal{M} (such as modular invariance for example) that result in a bounded $\varphi(\mathcal{K})$ of finite volume, the integral may give a nonzero result due to boundary terms. The energy per unit volume is alternatively expressed as

$$E = 2\pi\chi , \quad (2.16)$$

where χ is the Euler characteristic of \mathcal{K} .

2.1.1 Stringy cosmic strings

An example of particular importance, which will serve as the prototype of the solutions to be constructed later on, refers to the case where the moduli space consists of a single toroidal modulus $\tau = \tau_1 + i\tau_2$, such as the type IIB axion-dilaton or the complex-structure or Kähler modulus of an internal compactification torus. In this case, the scalar manifold is the $SL(2, \mathbb{R})/U(1)$ space, which is a Kähler manifold with Kähler potential

$$K = -\ln(i(\tau - \bar{\tau})) , \quad (2.17)$$

so that the effective action (2.1) takes the explicit form

$$S = \int d^D x \sqrt{-g} M_*^{D-2} \left(\frac{1}{2} R + \frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{(\tau - \bar{\tau})^2} \right) . \quad (2.18)$$

The solution to the equations of motion for the above action as we have seen are

$$\tau = \tau(z) , \quad ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \tau_2 |F(z)|^2 dz d\bar{z} , \quad (2.19)$$

whose energy per unit volume reads

$$E = -\frac{i}{2} \int_{\mathcal{K}} d^2 z \partial \bar{\partial} \ln \tau_2 . \quad (2.20)$$

To discuss the possible choices of $\tau(z)$, we first note that the effective action (2.15) has a symmetry under the $SL(2, \mathbb{R})$ group, acting as

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} , \quad ad - bc = 1 . \quad (2.21)$$

In the full theory, this symmetry group is broken to the modular group $PSL(2, \mathbb{Z})$, which is interpreted as a local symmetry. As a result, the space of inequivalent choices of the modulus τ is the quotient of the complex τ plane by the $PSL(2, \mathbb{Z})$ group, which can be taken to be the fundamental domain \mathcal{F}_1 specified by the conditions

$$|\tau_1| \leq \frac{1}{2} , \quad \tau_2 > 0 , \quad |\tau| \geq 1 , \quad \tau_1 \leq 0 \text{ if } |\tau| = 1 . \quad (2.22)$$

The above considerations imply that an arbitrary holomorphic ansatz for $\tau(z)$ is generally inconsistent since τ is restricted to live on \mathcal{F}_1 while z covers the whole Riemann sphere; this is alternatively verified by noting that for such a naive choice, e.g. $\tau(z) = z^n$, the energy per unit volume diverges. To construct consistent, finite-energy solutions we need a holomorphic function that provides a one-to-one mapping fundamental domain \mathcal{F}_1 to the Riemann sphere. This mapping is provided by the modular function $j(\tau)$ or, equivalently, by Klein's absolute invariant

$$J(\tau) = j(\tau) - 744 = \frac{41E_4(\tau)^3 + 31E_6(\tau)^2}{72\Delta(\tau)} , \quad (2.23)$$

where $E_4(\tau)$, $E_6(\tau)$ are the Eisenstein series of weight 4 and 6 respectively, and $\Delta(\tau) = (E_4(\tau)^3 - E_6(\tau)^2)/1728$ is the cusp form of weight twelve. The modular invariant has the asymptotic behavior

$$J(\tau) \sim q^{-1} , \quad \text{as } \tau_2 \rightarrow \infty . \quad (2.24)$$

Finite-energy solutions can then be constructed by equating $J(\tau(z))$ to a holomorphic function of z . Doing so, pulling back the integral in (2.20) from the z -plane to \mathcal{F}_1 , and converting it into a line integral over the boundary of \mathcal{F}_1 , we indeed obtain the finite expression

$$E = -\frac{iN}{2} \int_{\mathcal{F}_1} d^2\tau \partial_\tau \partial_{\bar{\tau}} \ln \tau_2 = \frac{\pi}{6} N , \quad (2.25)$$

where N is the number of times the z -plane covers \mathcal{F}_1 . Here, we will consider the rational maps

$$J(\tau(z)) = \frac{P(z)}{Q(z)} , \quad (2.26)$$

where $P(z)$ and $Q(z)$ are polynomials in z of degrees p and q respectively. For this choice, the integer N is equal to q if $p \leq q$ (in which case $J(\tau)$ approaches a constant value at infinity and diverges as $(z - z_i)^{-1}$ at the zeros of $Q(z)$ which are identified with the “cores” of the solutions) and equal to p if $p > q$ (in which case $J(\tau)$ diverges as z^{p-q} at infinity). In what follows we will consider the $p < q$ case where $J = b/(z - z_i)$. Given this ansatz for $\tau(z)$, it remains to choose the function $F(z)$ in (2.19) in such a way that the metric is modular-invariant and non-degenerate. The first requirement is fulfilled by noting that the $PSL(2, \mathbb{Z})$ transformation of τ_2 is given by

$$\tau_2 \rightarrow \frac{\tau_2}{|c\tau + d|^2} , \quad (2.27)$$

and hence can be compensated by multiplying τ_2 by $|f(\tau)|^2$ where $f(\tau)$ is an $PSL(2, \mathbb{Z})$ modular form of weight 1. This modular form is explicitly given in terms of the square of Dedekind eta function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = \Delta^{1/24} , \quad q = e^{2\pi i \tau} . \quad (2.28)$$

So the combination $\tau_2 |\eta(\tau)|^4$ is modular invariant. To fulfil the second requirement we note that, near the zeros z_i , we can write $q^{-1} \sim J(\tau) \sim (z - z_i)^{-1}$ which implies from (2.28) that $\tau_2 |\eta(\tau)|^4 \sim |(z - z_i)^{1/12}|^2$. Therefore, the choice $F(z) = \eta(\tau)^2 \prod_{i=1}^N (z - z_i)^{-1/12}$ leads to the modular-invariant, non-degenerate solution

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \tau_2 |\eta(\tau)|^4 \left| \prod_{i=1}^N (z - z_i)^{-1/12} \right|^2 dz d\bar{z} , \quad J(\tau(z)) = \frac{P(z)}{Q(z)} . \quad (2.29)$$

As $|z| \rightarrow \infty$, the internal metric approaches $k_{z\bar{z}} \sim (z\bar{z})^{-N/12}$. Hence, by standard arguments, the solution has a deficit angle $\delta = \frac{\pi}{6} N$, which is equal to the energy as expected. So, at infinity, the internal space has conical singularities which signify the geometry of a non-compact space. For $N = 12$ the internal space is cylindrical, while for $N = 24$ the internal space compactifies to \mathbf{S}^2 .

The above considerations are quite general and apply in all cases where toroidal moduli with $PSL(2, \mathbb{Z})$ symmetry exist. For instance, we can consider the case of $D = 4$ theories arising e.g. from string compactifications on an internal space containing a \mathbf{T}^2 . Then the resulting solution (2.29) corresponds to a configuration of N strings carrying charge under the toroidal modulus field. This solution is the stringy cosmic string of [28].

There is also the possibility to add $(D - 3)$ -brane sources at the points z_i where the scalar field diverges. In this case, an energy-momentum tensor of the form

$$T_{\mu\nu} = -\eta_{\mu\nu} \sum_{i=1}^N T_i \delta^{(2)}(z - z_i), \quad T_{z\bar{z}} = 0, \quad (2.30)$$

should be introduced, where T_i is the tension of the brane located at the point z_i . These branes cause additional deficit angles equal to their tensions. Then, from (2.14) we find that the modular-invariant metric is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \tau_2 |\eta(\tau)|^4 \left| \prod_{i=1}^N (z - z_i)^{-1/12} \right|^2 \left| \prod_{i=1}^N (z - z_i)^{-T_i/2\pi M_*^{D-2}} \right|^2 dz d\bar{z}. \quad (2.31)$$

At infinity the total deficit angle turns out to be

$$\delta = \frac{\pi}{6} N + \sum_{i=1}^N \frac{T_i}{M_*^{D-2}}, \quad (2.32)$$

which is equal to the energy (2.16) as expected. In order for the internal space to compactify to \mathbf{S}^2 , the above deficit angle must be equal to 4π . This amounts to a fine-tuning condition on the brane tensions T_i , namely $\sum_i^N T_i = 4\pi M_*^{D-2} (1 - \frac{N}{24})$.

At the vicinity of each brane ($z \rightarrow z_i$), where $J(\tau) \rightarrow \infty$, the internal metric becomes $k_{z\bar{z}} \sim \tau_2 (z - z_i)^{-T_i/\pi M_*^{D-2}}$. Then, contracting (2.11) with $k^{z\bar{z}}$ one deduces that the Ricci scalar is not singular for $T_i > 2\pi M_*^{D-2}$. But the true condition for the absence of curvature singularities follows when the previous condition and Eq. (2.32) are both satisfied. These conditions restrict the number of branes to $N = 1$, in accordance with the result of [29]. Note that in the absence of the extra term involving the tension in $k_{z\bar{z}}$ there is a curvature singularity.

2.2 Complex dimension two

We next proceed to the case where the internal space has complex dimension two [34], which is relevant for seeking codimension-four brane solutions with an internal compact or non-compact space. Now, Eq. (2.9) takes the form

$$k_{1\bar{1}} k_{2\bar{2}} - k_{1\bar{2}} k_{2\bar{1}} = e^{-K} |F(z, w)|^2, \quad (2.33)$$

which is a highly nonlinear differential equation. Equation (2.33) is very difficult to be solved given an explicit form of the Kähler potential K of the scalar manifold and a general holomorphic ansatz for the fields $\varphi^i(z, w)$. However, for the special case where K decomposes as the sum

$$K(\varphi^i, \bar{\varphi}^{\bar{j}}) = K^{(1)}(\varphi^A, \bar{\varphi}^{\bar{A}}) + K^{(2)}(\varphi^B, \bar{\varphi}^{\bar{B}}) , \quad (2.34)$$

with the first term involving a subset $(\varphi^A, \bar{\varphi}^{\bar{A}})$ of the $(\varphi^i, \bar{\varphi}^{\bar{i}})$ and the second term involving the remaining fields $(\varphi^B, \bar{\varphi}^{\bar{B}})$, we can easily solve this equation by assuming an ansatz of the form

$$\varphi^A = \varphi^A(z) , \quad \varphi^B = \varphi^B(w) , \quad k = k^{(1)}(z, \bar{z}) + k^{(2)}(w, \bar{w}) , \quad (2.35)$$

where the φ^A and φ^B depend only on z and w and the metric is the sum of two terms involving (z, \bar{z}) and (w, \bar{w}) respectively. Writing also $F(z, w) = F^{(1)}(z)F^{(2)}(w)$, Eq. (2.33) simplifies to

$$k_{1\bar{1}}^{(1)} k_{2\bar{2}}^{(2)} = \left[e^{-K^{(1)}} |F^{(1)}(z)|^2 \right] \left[e^{-K^{(2)}} |F^{(2)}(w)|^2 \right] , \quad (2.36)$$

and is easily solved by taking $k_{1\bar{1}}^{(1)}$ and $k_{2\bar{2}}^{(2)}$ equal to the first and second terms in brackets respectively. The final solution, which generalizes (2.31) then reads

$$\begin{aligned} \varphi^A &= \varphi^A(z) , \quad \varphi^B = \varphi^B(w) , \\ ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu + e^{-K^{(1)}} |F^{(1)}(z)|^2 \left| \prod_{i=1}^{N_1} (z - z_i)^{-T_i^{(1)}/2\pi M_*^{D-2}} \right|^2 dz d\bar{z} + \\ &e^{-K^{(2)}} |F^{(2)}(w)|^2 \left| \prod_{j=1}^{N_2} (w - w_j)^{-T_j^{(2)}/2\pi M_*^{D-2}} \right|^2 dw d\bar{w} . \end{aligned} \quad (2.37)$$

We will discuss in section 4.3 the interpretation of such a solution.

3. Special Kähler and Grassmannian

The construction of solutions of the type described in the previous section carries over to more complicated moduli spaces. Here, we will construct solutions of this form for the cases where the classical moduli space is a special Kähler manifold of the form

$$\mathcal{SK}_{n+1} = \frac{SL(2, \mathbb{R})}{U(1)} \times \frac{SO(2, n)}{SO(2) \times SO(n)} , \quad (3.1)$$

or a Kähler manifold of the form

$$\mathcal{K}_n = \frac{SO(2, n)}{SO(2) \times SO(n)} . \quad (3.2)$$

In what follows, we will give a brief description of the geometry of these manifolds, using the formalism of special geometry, and we will state the corresponding Kähler potentials for the cases of interest.

The geometry of the special Kähler manifold \mathcal{SK}_{n+1} is completely specified by a holomorphic symplectic section [30, 31]

$$\Omega = \begin{pmatrix} X^I \\ F_I \end{pmatrix} , \quad (3.3)$$

in terms of which the Kähler potential is given by

$$K = -\ln(\mathrm{i}\langle\Omega|\bar{\Omega}\rangle) \equiv -\ln(\mathrm{i}(\bar{X}^I F_I - X^I \bar{F}_I)) . \quad (3.4)$$

In the above, X^I , $I = 0, \dots, n+1$ are a set of complex parameters, while F_I are usually specified as the derivatives of a holomorphic prepotential $F(X)$ with respect to the X^I . In the present case, it is convenient to employ the so-called symplectic gauge in which Ω is written as

$$\Omega = \begin{pmatrix} X^I \\ F_I \end{pmatrix} = \begin{pmatrix} X^I \\ S\eta_{IJ}X^J \end{pmatrix} , \quad (3.5)$$

where $\eta_{IJ} = \text{diag}(+1, +1, -1, \dots, -1)$ is the $SO(2, n)$ invariant metric and S parametrizes the $\frac{SL(2, \mathbb{R})}{U(1)}$ factor in the usual way. X^I parametrize the $\frac{SO(2, n)}{SO(2) \times SO(n)}$ factor and are required to satisfy the $SO(2, n)$ orthogonality condition

$$\eta_{IJ}X^IX^J = 0 . \quad (3.6)$$

Although this gauge choice makes it impossible to specify F_I by means of a prepotential, Eq. (3.4) for the Kähler potential is perfectly valid, leading to the result [35]

$$K = K_1 + K_2 , \quad (3.7)$$

where

$$K_1 = -\ln(S - \bar{S}) \quad (3.8)$$

is the standard Kähler potential for $\frac{SL(2, \mathbb{R})}{U(1)}$ and

$$K_2 = -\ln(\eta_{IJ}\bar{X}^IX^J) \quad (3.9)$$

is the Kähler potential of $\frac{SO(2, n)}{SO(2) \times SO(n)}$. The latter can be verified by parametrizing X^I in terms of the independent Calabi-Vesentini coordinates y^a , $a = 1, \dots, n$, according to

$$X^I(y) = \begin{pmatrix} \frac{1}{2}(1 + y^2) \\ \frac{\mathrm{i}}{2}(1 - y^2) \\ y^a \end{pmatrix} . \quad (3.10)$$

Then, it is straightforward to see that the familiar formula

$$K_2 = -\ln(1 - 2y^\dagger y + |y^2|^2) , \quad (3.11)$$

is recovered.

We are particular interested in Kähler manifolds of the form (3.2) with $n = 1, 2, 3$ for which the modular forms required to construct our solutions are explicitly known. In what follows, we give the explicit parametrizations of X^I in terms of supergravity fields and we state the corresponding Kähler potentials for these particular cases.

- $n = 1$. For this case, the $SO(2, 1; \mathbb{R})$ vector X^I is parametrized in terms of a single complex field T as [31]

$$X^I(T) = \begin{pmatrix} \frac{1}{\sqrt{2}}(1 - T^2) \\ -\sqrt{2}T \\ -\frac{1}{\sqrt{2}}(1 + T^2) \end{pmatrix} . \quad (3.12)$$

Inserting this into (3.9), we find the Kähler potential

$$K_2(T) = -2\ln(T - \bar{T}) . \quad (3.13)$$

- $n = 2$. For this case, the $SO(2, 2)$ vector X^I is parametrized in terms of two complex fields T and U as [31, 36]

$$X^I(T, U) = \begin{pmatrix} \frac{1}{\sqrt{2}}(1 - TU) \\ -\frac{1}{\sqrt{2}}(T + U) \\ -\frac{1}{\sqrt{2}}(1 + TU) \\ \frac{1}{\sqrt{2}}(T - U) \end{pmatrix} , \quad (3.14)$$

and the Kähler potential reads

$$K_2(T, U) = -\ln((T - \bar{T})(U - \bar{U})) . \quad (3.15)$$

- $n = 3$. Now, the $SO(2, 3)$ vector X^I can be parametrized in terms of three complex fields T , U and V as

$$X^I(T, U, V) = \begin{pmatrix} \frac{1}{\sqrt{2}}(1 - TU + V^2) \\ -\frac{1}{\sqrt{2}}(T + U) \\ -\frac{1}{\sqrt{2}}(1 + TU - V^2) \\ \frac{1}{\sqrt{2}}(T - U) \\ \sqrt{2}V \end{pmatrix} , \quad (3.16)$$

and the Kähler potential reads

$$K_2(T, U, V) = -\ln((T - \bar{T})(U - \bar{U}) - (V - \bar{V})^2) . \quad (3.17)$$

To summarize, the Kähler potential for the special Kähler manifolds (3.1) is given by $K = K_1 + K_2$ where K_1 is given in (3.8) and K_2 is given in (3.9), while the Kähler potential for the Kähler manifolds (3.2) is simply $K = K_2$. Explicit expressions for K_2 for the cases $n = 1$, $n = 2$ and $n = 3$ are given in Eqs. (3.13), (3.15) and (3.17) respectively.

4. Application to supergravity theories

We may apply the results of the previous sections to construct solutions in the context of supergravity theories where scalar Kähler manifolds of the sort discussed earlier appear. In particular, we will discuss two classes of solutions. The first class corresponds to stringy-cosmic-string solutions of $D = 4$ supergravities with $\mathcal{N} = 4$ or $\mathcal{N} = 2$ supersymmetry, arising from appropriate heterotic string compactifications. The theories under consideration possess modular symmetries that may be exploited to construct stringy cosmic string solutions according to the guidelines of section 2. Moreover, for these theories, the quantum corrections to the Kähler potential are under control and thus one can extend the classical solutions to solutions that are exact to all orders in perturbation theory. The second class of solutions corresponds to five-brane solutions of minimal $D = 8$, $\mathcal{N} = 1$ supergravity and four-dimensional intersections thereof.

4.1 String solutions in $D = 4$, $\mathcal{N} = 4$ supergravity

We first consider the case of the $\mathcal{N} = 4$ theories [32] arising from compactifications of the $E_8 \times E_8$ or $SO(32)$ heterotic string theories on $\mathbf{T}^4 \times \mathbf{T}^2$ or, equivalently, from compactifications of $\mathcal{N} = 1$ six-dimensional supergravity on \mathbf{T}^2 . For these models, the moduli space consists of three factors involving (i) the axion-dilaton S , (ii) the moduli T and U corresponding to the complex and Kähler structure moduli of \mathbf{T}^2 , and (iii) the moduli of \mathbf{T}^4 . In what follows, we will consider only the first two types of moduli, which parametrize the space

$$\mathcal{M} = \left(PSL(2, \mathbb{Z}) \setminus \frac{SL(2, \mathbb{R})}{U(1)} \right)_S \times \left(SO(2, 2; \mathbb{Z}) \setminus \frac{SO(2, 2)}{SO(2) \times SO(2)} \right)_{T, U}, \quad (4.1)$$

with the isomorphism $SO(2, 2; \mathbb{Z}) \cong PSL(2, \mathbb{Z}) \times PSL(2, \mathbb{Z})$ implying that the duality group is given by the product $PSL(2, \mathbb{Z})_S \times PSL(2, \mathbb{Z})_T \times PSL(2, \mathbb{Z})_U$. The moduli S , T and U are given in terms of six-dimensional fields as

$$S = \alpha + i e^{-2\phi}, \quad T = B_{45} + i \sqrt{\det g_{mn}}, \quad U = \frac{g_{45}}{g_{55}} + i \frac{\sqrt{\det g_{mn}}}{g_{55}}, \quad (4.2)$$

where ϕ and α are the dilaton and axion while g_{mn} and B_{45} is the metric and B -field on \mathbf{T}^2 . The effective action for these fields follows from the Kähler potential in Eq. (3.15), namely

$$K = -\ln(S - \bar{S}) - \ln(T - \bar{T}) - \ln(U - \bar{U}). \quad (4.3)$$

It is invariant under the duality group, as well as under string/string/string triality [37] which interchanges S , T and U .

The solution for the moduli for this case is readily obtained by taking S , T and U to be holomorphic functions restricted to the fundamental domains of $PSL(2, \mathbb{Z})_S$, $PSL(2, \mathbb{Z})_T$ and $PSL(2, \mathbb{Z})_U$, respectively, by relations of the form (2.26), and by inserting the Kähler potential (4.3) in Eq. (2.12). This leads to a stringy cosmic string solution with transverse metric

$$d\sigma^2 = S_2 T_2 U_2 |F(z)|^2 dz d\bar{z} . \quad (4.4)$$

To determine $F(z)$, we impose the requirements of modular invariance and non-degeneracy of the metric as before. The first requirement leads to a factor of $|\eta(S)\eta(T)\eta(U)|^4$ while the second requirement leads to a factor of $|(z - z_i)^{-1/12}|^2$ for each string. Letting N_S , N_T and N_U be the number of strings carrying charge with respect to the S , T and U moduli respectively, we finally find

$$d\sigma^2 = S_2 T_2 U_2 |\eta(S)\eta(T)\eta(U)|^4 \left| \prod_{i=1}^{N_S} \prod_{j=1}^{N_T} \prod_{k=1}^{N_U} ((z - z_i)(z - z_j)(z - z_k))^{-1/12} \right|^2 dz d\bar{z} . \quad (4.5)$$

Imposing string/string/string triality leads to $N_S = N_T = N_U = N$. As $|z| \rightarrow \infty$ for each string we have a deficit angle $\delta = \frac{\pi}{6}$, and the energy of the solution is $E = \frac{\pi}{6}(N_S + N_T + N_U) = \frac{\pi}{2}N$. Therefore, to compactify the transverse space to \mathbf{S}^2 , we need $N = 8$.

In the above we have assumed that each string is charged with respect to only a single modulus so $z_i \neq z_j \neq z_k$. However, string/string/string triality also allows us to consider “ STU -strings” that are charged under all three moduli. Such configurations may give rise to orbifold singularities on the transverse space; in order for this to occur, we need deficit angles of the form $\delta = 2\pi(n - 1)/n$ where $n > 1$ is an integer. To discuss this, we first write the transverse metric (for the case $N = 8$) as

$$d\sigma^2 = S_2 T_2 U_2 |\eta(S)\eta(T)\eta(U)|^4 \left| \prod_{i=1}^8 (z - z_i)^{-1/4} \right|^2 dz d\bar{z} . \quad (4.6)$$

For the generic case where the locations z_i of the strings are different, we have a deficit angle of $\pi/2$ for each string and hence no orbifold singularities occur. However, when some of the z_i are identified, such singularities appear. For example, consider the case where the eight z_i coalesce into three points z_1 , z_2 and z_3 , of orders three, three and two respectively. Then the transverse metric turns to be

$$d\sigma^2 = S_2 T_2 U_2 |\eta(S)\eta(T)\eta(U)|^4 |(z - z_1)^{-3/4}(z - z_2)^{-3/4}(z - z_3)^{-1/2}|^2 dz d\bar{z} , \quad (4.7)$$

and one recognizes the deficit angles of $3\pi/2$, $3\pi/2$ and π around z_1 , z_2 and z_3 respectively. The transverse space is thus a $\mathbf{T}^2/\mathbf{Z}_4$ orbifold as we can see from δ for the $n = 4$ value. Another example is obtained by taking the eight z_i to coalesce into four points z_1, \dots, z_4 , of order two each. Then the transverse metric turns to

$$d\sigma^2 = S_2 T_2 U_2 |\eta(S)\eta(T)\eta(U)|^4 \left| \prod_{i=1}^4 (z - z_i)^{-1/2} \right|^2 dz d\bar{z}, \quad (4.8)$$

and one recognizes a deficit angle of π for each string. The transverse space is thus a $\mathbf{T}^2/\mathbf{Z}_2$ orbifold.

We may now consider sting sources located at the points z_i, z_j, z_k where the scalar fields S, T, U diverge with energy-momentum tensors of the form

$$T_{\mu\nu} = -\eta_{\mu\nu} \left(\sum_{i=1}^{N_S} T_i \delta^{(2)}(z - z_i) + \sum_{j=1}^{N_T} T_j \delta^{(2)}(z - z_j) + \sum_{k=1}^{N_U} T_k \delta^{(2)}(z - z_k) \right), \quad T_{z\bar{z}} = 0. \quad (4.9)$$

In this case, the solution (4.5) changes to

$$d\sigma^2 = S_2 T_2 U_2 |\eta(S)\eta(T)\eta(U)|^4 \left| \prod_{i=1}^{N_S} \prod_{j=1}^{N_T} \prod_{k=1}^{N_U} ((z - z_i)(z - z_j)(z - z_k))^{-1/12} \right|^2 \\ \times \left| \prod_{i=1}^{N_S} \prod_{j=1}^{N_T} \prod_{k=1}^{N_U} (z - z_i)^{-T_i/2\pi M_*^2} (z - z_j)^{-T_j/2\pi M_*^2} (z - z_k)^{-T_k/2\pi M_*^2} \right|^2 dz d\bar{z} \quad (4.10)$$

So at infinity the transverse space compactifies to \mathbf{S}^2 , if

$$4\pi = \frac{\pi}{6} (N_S + N_T + N_U) + \sum_{i=1}^{N_S} \frac{T_i}{M_*^2} + \sum_{j=1}^{N_T} \frac{T_j}{M_*^2} + \sum_{k=1}^{N_U} \frac{T_k}{M_*^2}. \quad (4.11)$$

Note that by imposing string/string/string triality we are led again to take $N_S = N_T = N_U = N$ in (4.11). Then again there are no curvature singularities when $N_S = N_T = N_U = 1$ and $T > 2\pi M_*^2$.

4.2 String solutions in $D = 4$, $\mathcal{N} = 2$ supergravities

We next consider the case of the $\mathcal{N} = 2$ theories arising from compactifications of heterotic string theories on $K3 \times \mathbf{T}^2$ or, equivalently, from compactifications of minimal $\mathcal{N} = 1$ six-dimensional supergravity coupled to vector multiplets on \mathbf{T}^2 [37]. For these models, the moduli space consists of (i) the vector-multiplet moduli space \mathcal{M}_V parametrized by the axion-dilaton S , the moduli T and U corresponding to combinations of the complex and Kähler structure moduli T^i of \mathbf{T}^2 , and the Wilson line moduli V^a , and (ii) the hypermultiplet

moduli space \mathcal{M}_H parametrized by the moduli of K_3 and of the vector bundle. Restricting to the vector multiplet moduli space, its classical geometry is locally of the form

$$\mathcal{M}_V = \left(PSL(2, \mathbb{Z}) \setminus \frac{SL(2, \mathbb{R})}{U(1)} \right) \times \left(SO(2, n; \mathbb{Z}) \setminus \frac{SO(2, n)}{SO(2) \times SO(n)} \right), \quad (4.12)$$

where $n = p + 2$ with p being the number of Wilson line moduli. Here, the $PSL(2, \mathbb{Z})$ and $SO(2, n; \mathbb{Z})$ are the S- and T-duality groups [38]. This moduli space falls into the class of special Kähler manifolds, considered in section 3.

For the construction of solutions of interest in the models considered here, there are two points that need special attention. First, the $PSL(2, \mathbb{Z})$ S-duality is no longer expected to be a symmetry of the full quantum theory and so consistent solutions can be constructed only by fixing the S modulus to a constant value and demanding invariance only under the T-duality group. Second, the prepotential is renormalized both perturbatively and nonperturbatively, where the $\mathcal{N} = 2$ non-renormalization theorems guarantee that the perturbative corrections enter only at one-loop order. Perturbatively exact solutions can thus be constructed by taking account of the one-loop corrections which, at the level of the Kähler potential, amount to the shift

$$S_2 \rightarrow S_2 + V_{GS}(T^i, V^a), \quad (4.13)$$

where $V_{GS}(T^i, V^a)$ is the Green-Schwarz term. Note that S and $V_{GS}(T^i, V^a)$ transform under T-duality in such a way that the corresponding transformation of K is a Kähler transformation. Given these observations, we may proceed to construct stringy cosmic string solutions for the special cases $n = 1, 2, 3$ where the modular forms used for the construction of invariant solutions are explicitly known.

4.2.1 The $n = 1$ ST model

The ST model corresponds to the case where Wilson line moduli are absent and only the T modulus of the torus is turned on [39]. It is obtained from the general case by setting $n = 1$. The T-duality group is then

$$SO(2, 1; \mathbb{Z}) \cong PSL(2, \mathbb{Z}), \quad (4.14)$$

and the classical Kähler potential is read off from Eq. (3.13),(3.8)

$$K(S, T) = -\ln(S - \bar{S}) - 2\ln(T - \bar{T}). \quad (4.15)$$

In the quantum theory, the above relation is modified by setting $S_2 \rightarrow S_2 + V_{GS}(T)$.

As remarked earlier on, the stringy cosmic string solutions of interest are constructed by fixing the S modulus to some constant value and imposing invariance under $PSL(2, \mathbb{Z})_T$ and non-degeneracy of the metric. The former requirement now leads to a factor of $|\eta(T)|^8$ while

the second requirement leads to a factor of $|(z - z_i)^{-1/6}|^2$ for each string (the different powers are due to the factor of two appearing in the Kähler potential). Therefore, our solution for the transverse metric reads

$$ds^2 = -dt^2 + dx^2 + (S_2 + V_{GS})T_2^2 |\eta(T)|^8 \left| \prod_{i=1}^N (z - z_i)^{-1/6} \right|^2 dz d\bar{z} . \quad (4.16)$$

At infinity for each string we have a deficit angle $\delta = \frac{\pi}{3}$, and the total energy is $E = \frac{\pi}{3}N$, i.e. the energy per string is twice that of the $\mathcal{N} = 4$ solution. The generalized solution becomes as (2.31).

4.2.2 The $n = 2$ STU model

The STU model corresponds to the case where Wilson line moduli are absent and both moduli of the torus are turned on [40, 37]. It is obtained by the general case by setting $n = 2$. The classical T-duality group is in this case

$$SO(2, 2; \mathbb{Z}) \cong PSL(2, \mathbb{Z})_T \times PSL(2, \mathbb{Z})_U , \quad (4.17)$$

and the classical Kähler potential is read off from Eq. (3.15), (3.8)

$$K(S, T, U) = -\ln(S - \bar{S}) - \ln((T - \bar{T})(U - \bar{U})) . \quad (4.18)$$

There is also a \mathbf{Z}_2 symmetry corresponding to the exchange $T \leftrightarrow U$. In the quantum theory, Eq. (4.18) is similarly modified by setting $S_2 \rightarrow S_2 + V_{GS}(T, U)$, while the \mathbf{Z}_2 symmetry mentioned above is broken.

The stringy cosmic string solution is constructed as before, and the result for the transverse metric is

$$d\sigma^2 = (S_2 + V_{GS})T_2 U_2 |\eta(T)\eta(U)|^4 \left| \prod_{i=1}^{N_T} \prod_{j=1}^{N_U} (z - z_i)^{-1/12} (z - z_j)^{-1/12} \right|^2 dz d\bar{z} . \quad (4.19)$$

As $|z| \rightarrow \infty$ for each string we have a deficit angle $\delta = \frac{\pi}{6}$, the total energy is $E = \frac{\pi}{6}(N_T + N_U)$ and due to the fact that the \mathbf{Z}_2 exchange symmetry is broken, the numbers N_T and N_U may be different. Regularity of the solution requires $N_T + N_U = 24$. The generalized solution becomes as in (4.10) with the factors corresponding to the N_S strings omitted.

4.2.3 The $n = 3$ $STUV$ model

The final case we will consider here is the $STUV$ model [41], which corresponds to turning on a single Wilson line modulus in addition to the two moduli of the torus. It is obtained by the general case by setting $n = 3$ so that classical T-duality group is

$$SO(2, 3; \mathbb{Z}) \cong Sp(4, \mathbb{Z}) . \quad (4.20)$$

In the genus two case the moduli space is the quotient of the Siegel upper half space by the modular group $PSp(4, \mathbb{Z})$, which can be taken to be the fundamental domain \mathcal{F}_2 , and is parametrized by the period matrix Ω , which transforms according to $\Omega \rightarrow (A\Omega + B)(C\Omega + D)^{-1}$. This matrix is specified as

$$\Omega = \begin{pmatrix} T & V \\ V & U \end{pmatrix} . \quad (4.21)$$

A Siegel modular form F_w of weight w is defined as a holomorphic function of Ω that transforms as

$$F_w(\Omega) \rightarrow (\det(C\Omega + D))^w F_w(\Omega) . \quad (4.22)$$

Any such form admits a Laurent expansion in the parameters $q = e^{2\pi i T}$, $r = e^{2\pi i V}$ and $s = e^{2\pi i U}$. The graded ring of Siegel modular forms is generated [42, 43] by four forms of weight 4, 6, 10 and 12, namely by the two Eisenstein series ψ_4 and ψ_6 and the two cusp forms χ_{10} and χ_{12} . In the degeneration limit $\epsilon \rightarrow 0$, where $Sp(4, \mathbb{Z})$ degenerates to $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$, the genus two surface can be constructed from two tori with modular parameters $q_1 = e^{2\pi i \tilde{T}}$ and $q_2 = e^{2\pi i \tilde{U}}$ [44, 45]. These two tori are joined by excising a disk of radius $|\epsilon|$ from each torus and making an appropriate identification of two annular regions around the excised disk. In this limit the relations between the parameters T, U, V and $\tilde{T}, \tilde{U}, \epsilon$ are as follows [45]

$$T = \tilde{T} + \mathcal{O}(\epsilon^2) \quad U = \tilde{U} + \mathcal{O}(\epsilon^2) \quad V = -\epsilon + \mathcal{O}(\epsilon^3) . \quad (4.23)$$

Turning now to the classical Kähler potential, this is read off Eq. (3.17), (3.8)

$$\begin{aligned} K(S, T, U, V) &= -\ln(S - \bar{S}) - \ln((T - \bar{T})(U - \bar{U}) - (V - \bar{V})^2) \\ &= -\ln(S - \bar{S}) - \ln \det(\Omega - \bar{\Omega}) , \end{aligned} \quad (4.24)$$

Again, in the quantum theory, Eq. (4.24) is modified by setting $S_2 \rightarrow S_2 + V_{GS}(T, U, V)$.

The stringy cosmic string solution for the model under consideration is obtained by generalizing the standard procedure to the $Sp(4, \mathbb{Z})$ case. First, the space of inequivalent choices for Ω is, as said, the fundamental domain \mathcal{F}_2 specified by the conditions

$$\begin{aligned} |T_1|, |U_1|, |V_1| &\leq \frac{1}{2} , \quad 0 \leq |2V_2| \leq T_2 \leq U_2 , \\ |\det(C\Omega + D)| &\geq 1 \text{ for all } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4, \mathbb{Z}) . \end{aligned} \quad (4.25)$$

To construct finite-energy solutions, we need a set of holomorphic functions that provide a map from the variable Ω , which is restricted to live on \mathcal{F}_2 according to (4.25), to the Riemann

sphere, i.e. the $Sp(4, \mathbb{Z})$ counterparts of the J -function. Such functions exist (known as Igusa invariants [46]) and are explicitly given in terms of the $Sp(4, \mathbb{Z})$ Eisenstein series ψ_4 , ψ_6 and the cusp forms χ_{10} , χ_{12} as follows

$$x_1 = \frac{\psi_4 \chi_{10}^2}{\chi_{12}^2}, \quad x_2 = \frac{\psi_6 \chi_{10}^3}{\chi_{12}^3}, \quad x_3 = \frac{\chi_{10}^6}{\chi_{12}^5}. \quad (4.26)$$

Using the Kähler potential (4.24), we find the transverse metric

$$d\sigma^2 = (S_2 + V_{GS}) \det \Omega_2 |F(z)|^2 dz d\bar{z}, \quad (4.27)$$

where Ω_2 equals to $\text{Im } \Omega$ and now the function $F(z)$ must be chosen so as to enforce $Sp(4, \mathbb{Z})$ modular invariance and non-degeneracy of the metric. To ensure modular invariance, we note that the $Sp(4, \mathbb{Z})$ transformation of $\det \Omega_2$ reads

$$\det \Omega_2 \rightarrow \frac{\det \Omega_2}{|\det(C\Omega + D)|^2} \quad (4.28)$$

and hence can be compensated by multiplying by $|f(\Omega)|^2$, where $f(\Omega)$ is an $Sp(4, \mathbb{Z})$ modular form of weight 1 as follows from (4.22) with no zeros on the fundamental domain \mathcal{F}_2 . The unique form with these properties is given by the twelfth root of the cusp form χ_{12} , i.e. $f(\Omega) = \chi_{12}^{1/12}(\Omega)$. In order to have non-degenerate metric, we note that the poles of Igusa invariants are determined by the zeros of the cusp form χ_{12} , as one can see from Eq.(4.26). This cusp form has zeros in the z -plane at the locus $q = s = 0$, where the locations of the T -, U - string cores z_i , z_j exist. As we go around such a string, Ω should undergo an $Sp(4, \mathbb{Z})$ transformation generated by the $Sp(4, \mathbb{Z})$ matrices

$$T_i = \begin{pmatrix} 1_{2 \times 2} & s_i \\ 0 & 1_{2 \times 2} \end{pmatrix}, \quad (4.29)$$

where

$$s_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (4.30)$$

This leads to the $Sp(4, \mathbb{Z})$ jumps $\Omega \rightarrow \Omega + s_i$, or in terms of T, U, V ,

$$T \rightarrow T + 1, \quad U \rightarrow U + 1, \quad V \rightarrow V + 1. \quad (4.31)$$

These monodromies and holomorphicity require that near the core of the string, we will have

$$T \sim \frac{1}{2\pi i} \ln(z - z_i), \quad U \sim \frac{1}{2\pi i} \ln(z - z_j), \quad V \sim \frac{1}{2\pi i} \ln(z - z_k), \quad (4.32)$$

so that

$$q = e^{2\pi i T} \sim (z - z_i), \quad s = e^{2\pi i U} \sim (z - z_j), \quad r = e^{2\pi i V} \sim (z - z_k). \quad (4.33)$$

Note that, due to (4.25), V should degenerate together with T and/or U , i.e, z_k should coincide with z_i and/or z_j .

Turning to χ_{12} , its full expansion is given in [44], which to leading order reads

$$\chi_{12} = 96qs + \dots \quad (4.34)$$

Then, from (4.33) and (4.34) follows that the form of $F(z)$ in the transverse metric is determined to be $F(z) = \chi_{12}^{1/12} \prod_{i=1}^{N_T} \prod_{j=1}^{N_U} (z - z_i)^{-1/12} (z - z_j)^{-1/12}$. This leads to the modular invariant, non-degenerate solution

$$d\sigma^2 = (S_2 + V_{GS}) \det \Omega_2 |\chi_{12}|^{1/6} \left| \prod_{i=1}^{N_T} \prod_{j=1}^{N_U} (z - z_i)^{-1/12} (z - z_j)^{-1/12} \right|^2 dz d\bar{z} \quad (4.35)$$

As $|z| \rightarrow \infty$ for each string the deficit angle is $\delta = \frac{\pi}{6}$ and the energy is indeed finite,

$$E = \frac{\pi}{6} (N_T + N_U) \quad (4.36)$$

Regularity of the solution demands that $N_T + N_U = 24$.

In the degeneration limit $\epsilon \rightarrow 0$, $Sp(4, \mathbb{Z})$ degenerates to $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$. In this limit, the Eisenstein series ψ_4 , ψ_6 and the cusp forms χ_{10} , χ_{12} take the form

$$\begin{aligned} \psi_4 &= E_4(q_1)E_4(q_2) + \mathcal{O}(\epsilon^2) \\ \psi_6 &= E_6(q_1)E_6(q_2) + \mathcal{O}(\epsilon^2) \\ \chi_{10} &= \epsilon^2 \Delta(q_1)\Delta(q_2) + \mathcal{O}(\epsilon^4) \\ \chi_{12} &= \Delta(q_1)\Delta(q_2) + \mathcal{O}(\epsilon^2) \end{aligned} \quad (4.37)$$

where $E_4(q_i)$, $E_6(q_i)$, $\Delta(q_i)$, $i=1,2$ are the weight 4 and 6 Eisenstein series and the cusp form of weight 12 respectively for each $SL(2, \mathbb{Z})$ factor. Then a linear combination of Igusa invariants x_1, x_2, x_3 , gives again a modular invariant form. In particular, using the linear combination

$$\frac{\alpha(x_1)^3 + \beta(x_2)^2 - \gamma x_3}{x_3} = \frac{\alpha(\psi_4)^3 + \beta(\psi_6)^2 - \gamma \chi_{12}}{\chi_{12}} \quad (4.38)$$

where $\alpha = \frac{41}{72}$, $\beta = \frac{31}{72}$, $\gamma = 732096$ and substituting the expressions (4.37) for ψ_4 , ψ_6 , χ_{12} , which are valid in the limit $\epsilon \rightarrow 0$, leads to the modular invariant form $J(q_1)J(q_2)$ corresponding to the $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ case.

Now the zeros of the cusp form χ_{12} are easily found, as for $\epsilon \rightarrow 0$ $\chi_{12} \rightarrow \Delta(q_1)\Delta(q_2)$, so that $\chi_{12} \rightarrow 0$ for $(q_1, \epsilon) \rightarrow (0, 0)$ and $(q_2, \epsilon) \rightarrow (0, 0)$. This implies that near the zeros z_i and z_j , we can write $\det \Omega_2 |\chi_{12}^{1/12}|^2 \sim |(z - z_i)^{1/12} (z - z_j)^{1/12}|^2$. Therefore, the appropriate choice for $F(z)$ is $F(z) = \chi_{12}^{1/12} \prod_{i=1}^{N_{\tilde{T}}} \prod_{j=1}^{N_{\tilde{U}}} (z - z_i)^{-1/12} (z - z_j)^{-1/12}$. Then using the fact that $\chi_{12} \rightarrow \Delta(q_1)\Delta(q_2)$ and $\det \Omega_2 = T_2 U_2$, one recovers the solution of the STU model appeared in the previous section.

4.3 Brane solutions in $D = 8$, $\mathcal{N} = 1$ supergravity

Another situation where Kähler manifolds of the type $\mathcal{K}_n = \frac{SO(2,n)}{SO(2) \times SO(n)}$ examined in section 3 occur is $\mathcal{N} = 1$ supergravity coupled to n vector multiplets in eight dimensions [33]. Each vector multiplet contains 2 scalars so that the total $2n$ scalars parametrize the coset \mathcal{K}_n . For this case, we can construct codimension-two solutions for $n = 1, 2, 3$ corresponding to five-branes. The $n = 1, 2$ cases also appear as solutions to minimal $D = 9$ and $D = 7$ supergravities [47] coupled to two vector multiplets.

Starting from codimension-two solutions, these can be constructed by considering the $\frac{SO(2,n)}{SO(2) \times SO(n)}$ Kähler potential (K_2 in the notation of section 3). The resulting transverse metrics are readily obtained from those of section 4.2 by simply discarding the S modulus. Therefore, for the case $n = 1$ where there exists a single modulus T , we obtain the solution

$$ds^2 = -dt^2 + dx_5^2 + T_2^2 |\eta(T)|^8 \left| \prod_{i=1}^N (z - z_i)^{-1/6} \right|^2 dz d\bar{z} , \quad (4.39)$$

where we will denote by dx_p^2 the spatial metric on the world-volume of a p-brane. The generalized solution is like (2.31). In the $D = 9$, $N = 1$ supergravity coupled to n vector multiplets the scalars parametrize the coset $SO(1, n)/SO(n)$. It is clear that for two vector multiplets coupled to gravity the codimension-two solution is like (4.39).

For the case $n = 2$ where there exist two moduli T and U , we find

$$ds^2 = -dt^2 + dx_5^2 + T_2 U_2 |\eta(T)\eta(U)|^4 \left| \prod_{i=1}^{N_T} \prod_{j=1}^{N_U} (z - z_i)^{-1/12} (z - z_j)^{-1/12} \right|^2 dz d\bar{z} . \quad (4.40)$$

Finally, for the case $n = 3$ where there exist the three moduli T , U and V , we have

$$ds^2 = -dt^2 + dx_5^2 + \det \Omega_2 |\chi_{12}|^{1/6} \left| \prod_{i=1}^{N_T} \prod_{j=1}^{N_U} (z - z_i)^{-1/12} (z - z_j)^{-1/12} \right|^2 dz d\bar{z} . \quad (4.41)$$

The generalized solution is as (4.10) by discarding the N_S strings. In the $D = 7$, $N = 2$ supergravity coupled to n vector multiplets, $3n$ scalars parametrize the coset $SO(3, n)/SO(3) \times SO(n)$. When the number of vector multiplets is two then the codimension-two solution is like (4.41).

Turning to codimension-four solutions, these can be constructed according to guidelines of section 2.2. The simplest possible situation is when the Kähler potential K decomposes as in Eq. (2.34) and is realized when $n = 2$, in which case we have

$$K(T, U) = K^{(1)}(T) + K^{(2)}(U) = -\ln(T - \bar{T}) - \ln(U - \bar{U}) . \quad (4.42)$$

Then, setting $T = T(z)$ and $U = U(w)$ we obtain $k_{1\bar{1}} = T_2 |F^{(1)}(z)|^2$ and $k_{2\bar{2}} = U_2 |F^{(2)}(w)|^2$. Determining the functions $F^{(1)}(z)$ and $F^{(2)}(w)$ in the usual manner, we finally obtain the metric (2.37), in the presence of tensions

$$ds^2 = -dt^2 + dx_3^2 + T_2 |\eta(T)|^4 \left| \prod_{i=1}^{N_T} (z - z_i)^{-1/12} \right|^2 \left| \prod_{i=1}^{N_T} (z - z_i)^{-T_i^{(1)}/2\pi M_*^6} \right|^2 dz d\bar{z} \\ + U_2 |\eta(U)|^4 \left| \prod_{j=1}^{N_U} (w - w_j)^{-1/12} \right|^2 \left| \prod_{j=1}^{N_U} (w - w_j)^{-T_j^{(2)}/2\pi M_*^6} \right|^2 dw d\bar{w} , \quad (4.43)$$

The deficit angles at infinity in the z, w -plane are

$$\delta_1 = \frac{\pi}{6} N_T + \sum_{i=1}^{N_T} \frac{T_i^{(1)}}{M_*^6} , \quad \delta_2 = \frac{\pi}{6} N_U + \sum_{j=1}^{N_U} \frac{T_j^{(2)}}{M_*^6} , \quad (4.44)$$

respectively. With $\delta_i = 4\pi$, we get an $\mathbf{S}^2 \times \mathbf{S}^2$ compactification of the $D = 8, \mathcal{N} = 1$ supergravity.

We may easily interpret the solution (4.43) by calculating the corresponding energy momentum tensor T_{MN} . We may write $T_{MN} = T_{MN}^\sigma + \sum_i^{N_T} T_{1,MN}^{(i)} + \sum_j^{N_U} T_{2,MN}^{(j)}$, where T_{MN}^σ is the scalar energy-momentum tensor and $T_{MN}^{(i)}$ is the contribution of the brane located at the point z_i . Then, by going to real coordinates $z = x^4 + ix^5, w = x^6 + ix^7$ we find that

$$T_{1,\mu\nu}^{(i)} = -\eta_{\mu\nu} T_i^{(1)} \delta^{(2)}(z - z_i), \quad T_{1,mn}^{(i)} = -g_{mn} T_i^{(1)} \delta^{(2)}(z - z_i), \quad T_{1,rs}^{(i)} = 0, \\ T_{2,\mu\nu}^{(j)} = -\eta_{\mu\nu} T_j^{(2)} \delta^{(2)}(w - w_j), \quad T_{2,mn}^{(j)} = 0, \quad T_{2,rs}^{(j)} = -g_{rs} T_j^{(2)} \delta^{(2)}(w - w_j). \quad (4.45)$$

where $(m, n = 4, 5)$ and $(r, s = 6, 7)$ in (4.45) represents intersecting five-branes of tensions $T^{(1)}, T^{(2)}$ with world-volumes extended across the (012345) and (012367) directions. Their common world-volume in the (0123) direction is the 4D Minkowski intersection.

5. Conclusions

We have presented here codimension-two solutions of supergravity models in diverse dimensions, with or without brane sources. We have considered in particular D -dimensional supergravity theories coupled to a set of scalar fields forming a nonlinear sigma model targeted on some non-compact manifold. The scalar manifolds employed are special Kähler of the form $\frac{SL(2, \mathbb{R})}{U(1)} \times \frac{SO(2, n)}{SO(2) \times SO(n)}$ [30, 31, 32] or the Grassmannian cosets $\frac{SO(2, n)}{SO(2) \times SO(n)}$ [33]. The solutions we found are of the general form $M^{D-n} \times \mathcal{K}$, where M^{D-n} is a flat Minkowski space and \mathcal{K} is an n -dimensional internal space. We tried to keep the discussion as general as possible. However, for concreteness we have considered the cases of $\mathcal{N} = 4$ and $\mathcal{N} = 2$ supergravity in

4 dimensions as well as minimal supergravities coupled to vector multiplets in 8 dimensions. In the first case, the solution presents a string and the 4D spacetime is compactified down to two-dimensions by a number of such strings. In the former case, the solution presents a five-brane or intersecting five-branes along four-dimensional flat space, compactifying this way the eight-dimensional supergravity down to 4D Minkowski space-time.

The explicit solutions were found by employing a holomorphic ansatz for the scalars. The latter were restricted to lie in the fundamental domain of the modular groups and allowing modular $SL(2, \mathbb{Z})$, $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ or $Sp(4, \mathbb{Z})$ jumps around certain points in the internal space. This modular jumps permit scalar field configurations with finite energy per unit volume and explicit solutions presented only for those cases where the modular forms required to construct the solutions were explicitly known. Note that in order the solutions to have finite energy, one has to arrange the total deficit angle produced by the scalar configurations to be 4π in which case the internal space compactifies to \mathbf{S}^2 .

All the solutions we have described have be generalized to include brane configurations as well, with the only requirement that the scalars of the theory do not couple to the branes. The latter induces further deficit angles, proportional to their tensions in the internal space. The requirement for the absence of conical singularities may be fulfilled by suitably tuning the brane tensions so that the total deficit angle equals 4π and leading to a smooth sphere compactification. It should also be noted that configurations of this type might be relevant for the solution of the cosmological constant problem as the world-volume of the branes are always flat irrespectively of any bulk dynamics.

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